Getting used to **abstract Mathematical thinking** to enhance and improve work with **concrete data**

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Abstract vs Concrete
Walk smelling the flowers or daydreaming?

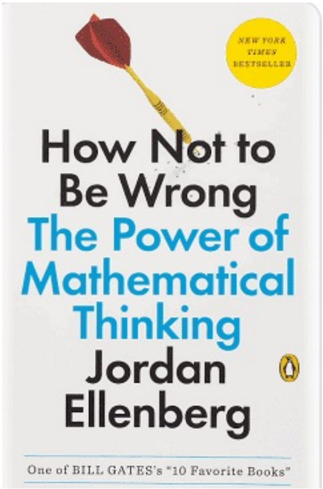
CONCRETE COGNITIVE REALITY:

- Being present: The here and now can be experienced through the senses. For example, when you are present in nature you can smell the flowers, feel the breeze, the warmth of the sum and the song of the birds.

ABSTRACT COGNITIVE REALITY:

- When you are not present: You are modeling the world – be cautious because
  - Every model is wrong, but some are useful (G. Box)
  - Every word we utter is a model
  - Every word is a lie, but some are useful (paraphrasing J. L. Borges)

Abstract (math) vs Concrete (experience)
What do people say about math?



"*How Not to Be Wrong* is a cheery manifesto for the utility of mathematical thinking. Ellenberg's prose is a delight—informal and robust, irreverent yet serious. **Maths is 'an atomic-powered prosthesis that you attach to your common sense, vastly multiplying its reach and strength,'** he writes. **Doing maths 'is to be, at once, touched by fire and bound by reason. Logic forms a narrow channel through which intuition flows with vastly augmented force.'"** —***The Guardian***

Abstract cognitive reality needs an embodiment to exist

Symbols perceived through the senses

- EYES:
  - Symbols in 1-D: Natural written Language, Formal Language.
  - Symbols in 2-D: Drawings (informal art or formal science).
- EARS:
  - Sounds: Informal visual arts, music, Natural spoken Language.
- SMELL and TASTE:
  - Aroma Therapy, Cuisine.
- TOUCH:
  - Language of touch.

Mathematical Structures

Abstract cognitive reality needs an embodiment to exist

- Mathematicians are concerned with formal language that creates a world that is stable, consistent and complete.

Mathematical structures are the building blocks of Mathematics

- Mathematical structure exists because of their properties, since they are not material.

Sets and Number systems (groups, rings and fields)

Sets represent attention to some items, concrete or abstract

- Sets cannot have repeated elements since every item in the real world is unique.
- Set is a concept, and it can be thought of, written down or spoken.
- In writing use curly brackets for sets: myset = {a,3,ç} is a set with 3 elements.
- Sets have two properties:
  - Cardinality: each set has a number of elements
  - Uniqueness or entries: no element can be repeated

Natural Numbers

The Natural numbers are born from the cardinality of sets.

- The empty set {} has 0 elements
- The set { {} } has 1 element (the empty set)

They are also called the Cardinal Numbers (0,1,2,3,…)

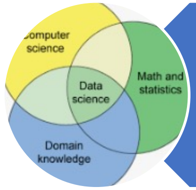
- https://en.wikipedia.org/wiki/Natural_number

The Natural Numbers have two operations:

- Addition (+): representing the cardinality of union of disjoint sets
- Multiplication (x): representing the repeated addition, e.g 2x3 means “repeat the addition of the number 3 with itself two times”, or 3+3

Step back and reflect:

Why are we developing Math from scratch?



In Data Science the meaning of Mathematics is as important as the methods of Mathematics

There is not enough emphasis in the meaning of Mathematics in the US educational system at the lower levels

Understanding the “Story Telling” behind Mathematics will make it easier to remember

Being familiar with the meaning of Mathematics will allow for creative work, which is essential in DS&A.

Creating an abstract algebra structure (the Integers)

- The Cardinal Numbers do not allow us to “reverse” the operation of addition (+). We can add 5 to 4, but we cannot remove 5 from 4.
- Let’s create new numbers by allowing to “reverse” the operation of addition removing can be denoted by “-”: removing 5 from 4 is denoted 4 – 5. The result is not a Cardinal Number. For obvious reasons we will call this new number -1 (“minus one”), and all the numbers with a “-” in front will be called “negative” numbers, while the rest (except 0) will be called “positive” numbers.
- The set of positive, 0 and negative numbers is called the Integers. See https://en.wikipedia.org/wiki/Integer

The Integers: properties and notation

Notation: Bold capital z (**Z**) or blackboard z ($\mathbb{Z}$).

- The set of integers with addition form a group (https://en.wikipedia.org/wiki/Group_(mathematics))
  - Closed
  - Have a neutral element with respect to addition
  - Every element has an inverse with respect to addition
  - Associativity (if you add three numbers, it does not matter which two you add first)

The set of integers with addition and multiplication form a ring with identity

- The set of integers form a ring (https://en.wikipedia.org/wiki/Ring_(mathematics))
- It is a group with respect to addition
- Multiplication is closed
- It has a neutral or identity element with respect to multiplication
- Multiplication is distributive with respect to addition

Other groups of interest in DS&A

Cyclic group (https://en.wikipedia.org/wiki/Cyclic_group)

- Denoted by Z/nZ of just Z_n
- The set of elements in Zn contains the integers 0,1,2…,(n-1) with operation of addition modulo n.

Example:

- Z_3 has 3 elements {0,1,2} with a_3+b_3 being the remainder of diving the sum of the integers a+b by 3. This is often called clock arithmetic.

Algebra vs Arithmetic

ARITHMETIC

- Arithmetic denotes all the work done with elements of a group or ring using only **constants**.
- Examples
  - in Z_5, $4_5+4_5 = 3_5$
  - in Z_n, $(n-1)_n + 1_n = 0_n$ (Note: “n” is a constant.)
- Notation: whenever it is clear, we will drop the subscripts from constants

ALGEBRA

- Algebra denotes all the work done with elements of a group or ring using **constants** and **variables.**
- Example
  - In Z_5: $4_5 + x = 3_5$ has only one solution. (Note: “x” is a variable.)
  - In Z_n: $(n-1)_n + x = 0_n$ has only one solution. Note: “x” is a variable but “n” is a constant.)

Equations as a way to generate information

Equations

- Writing symbolic expressions on each side of an equal sign forms an equation.
- Equations can be used for many purposes

Examples

- 2+2 = 5 is an equation that is FALSE (more about "FALSE" later)
- 2*x = 4 is an equation that has 1 solution in **Z.**
- 2*x = 5 is an equation that has no solutions in **Z**.

Equations as a way to generate fractions

Solutions to equations hint at how to generate new numbers

- $6x=3$ has as a solution a number that should cut in half 6 to get 3. Using the well known (and easy to explain; I will take it for granted that you know) technique, we divide both sides to get $x = 3/6$.
- The set of all fractions is denoted **Q**

Equations to generate Irrationals

More numbers!!!

- $X^2 = 2$ has as solutions numbers such that when each is multiplied by itself it produces 2. No integer or fraction can do that (not easy to prove at this level of math).
- $X^2 = 2$ has in fact two solutions: x=sqrt(2) or x=-sqrt(2)
- Sqrt(2) is an element of the Irrational numbers (numbers that cannot be written as an integer or fraction)

Equations to generate Complex numbers

One last set of numbers ☺, and no more.

- $x^2=-1$ has as a solution a number that when squared it gives -1. There is no Integer, Rational or Irrational number that does that.
- We denote the solution to $x^2=-1$ by i. In other words, the number denoted by the lower case letter i is such that $i^2=-1$. You could say that i = sqrt(-1).
- NOTE: The Real numbers consist of the union of Rationals and Irrationals.
- The set of Reals with the new number i and all the extra numbers to make it into a closed system is called the Complex numbers.

Storytelling, DSA and Mathematics

Exposure to the Foundations of Mathematics

- Watch in YouTube the video titled “Math Has a Fatal Flaw”:
  - https://www.youtube.com/watch?v=HeQX2HjkcNo

Collect data for storytelling while watching the video

- Write by hand on a blank sheet and bring it to class:
  - Mathematician names and what they did.
  - Salient facts and dates.